e^+e^- pair production in ultrarelativistic heavy-ion collisions at intermediate impact parameters

R.N. Lee* and A.I. Milstein†

Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia

(Dated: February 4, 2008)

Abstract

Using the quasiclassical Green's function in the Coulomb field, we analyze the probabilities of single and multiple e^+e^--pair production at fixed impact parameter b between colliding ultrarelativistic heavy nuclei. We perform calculations in the Born approximation with respect to the parameter $Z_B\alpha$, and exactly in $Z_A\alpha$, Z_A and Z_B are the charge numbers of the corresponding nuclei. We also obtain the approximate formulas for the probabilities valid for $Z_A\alpha$, $Z_B\alpha \lesssim 1$.

*Email:R.N.Lee@inp.nsk.su

†Email:A.I.Milstein@inp.nsk.su

I. INTRODUCTION

The cross section of e^+e^- pair production in ultrarelativistic heavy-ion collisions is very large, and this process can be a serious background for many experiments. Besides, it is also important for the problem of beam lifetime and luminosity of hadron colliders. It means that various corrections to the Born cross section for one-pair production, as well as the cross section for n-pair production ($n > 1$), are very important. Recently, the process was discussed in numerous papers , see reviews [1, 2, 3]. However, some important aspects of the problem have not been finally understood, and in the present paper we are going to elucidate them.

For our purpose, it is convenient to consider a collision of the nuclei A and B with the corresponding charge numbers Z_A and Z_B in the rest frame of the nucleus A. The nucleus B is assumed to move in the positive direction of the z axis having the Lorentz factor γ. For $\gamma \gg 1$, it is possible to treat the nuclei as sources of the external field, and calculate the probability of n-pair production $P_n(b)$ in collision of two nuclei at a fixed impact parameter b. The corresponding cross section σ_n is obtained by the integration over the impact parameter,

$$\sigma_n = \int d^2\, b\, P_n(b)\,. \tag{1}$$

Average number of the produced pairs at a given b reads:

$$W(b) = \sum_{n=1}^{\infty} n P_n(b)\,. \tag{2}$$

The function $W(b)$ defines the number-weighted cross section

$$\sigma_T = \int d^2\, b\, W(b) = \sum_{n=1}^{\infty} n\sigma_n\,. \tag{3}$$

The closed expression for σ_T was obtained in Refs. [4, 5, 6], though the correct meaning of this expression was recognized later in Ref. [7].

The cross section σ_T can be presented in the form:

$$\sigma_T = \sigma_T^0 + \sigma_T^C + \sigma_T^{CC}\,, \tag{4}$$

where σ_T^0 is the Born cross section, i.e., the cross section calculated in the lowest-order perturbation theory with respect to the parameters $Z_{A,B}\,\alpha$ ($\sigma_T^0 \propto (Z_B\alpha)^2(Z_A\alpha)^2$, $\alpha = e^2$ is the

fine-structure constant, e is the electron charge, $\hbar = c = 1$), σ_T^C is the Coulomb corrections with respect to one of the nuclei (containing the terms proportional to $(Z_B\alpha)^2(Z_A\alpha)^{2n}$ or $(Z_B\alpha)^{2n}(Z_A\alpha)^2$, $n \geqslant 2$), and σ_T^{CC} is the Coulomb corrections with respect to both nuclei (containing the terms proportional to $(Z_B\alpha)^n(Z_A\alpha)^l$ with $n, l > 2$). The cross section σ_T^0 coincides with the Born cross section of one pair production, which was calculated many years ago in Refs. [8, 9].

The expression for $W(b)$ derived in Refs. [4, 5, 6] requires regularization. The correct regularization was made in Refs. [10, 11], where the expressions for σ_T^C and σ_T^{CC} were obtained in the leading logarithmic approximation:

$$\begin{aligned} \sigma_T^C &= -\frac{28}{9\pi}\frac{\zeta}{m^2}\, L^2 \left[f(Z_B\alpha) + f(Z_A\alpha)\right], \\ \sigma_T^{CC} &= \frac{56}{9\pi}\frac{\zeta}{m^2}\, L\, f(Z_B\alpha) f(Z_A\alpha)\,, \\ \zeta = (Z_A\alpha)^2(Z_B\alpha)^2\,, \quad L &= \ln\gamma\,, \quad f(x) = \mathrm{Re}[\psi(1 + iZ_A\alpha) + C]\,, \end{aligned} \tag{5}$$

where m is the electron mass, $\psi(x) = \Gamma'(x)/\Gamma(x)$, and $C = 0.577\ldots$ is the Euler constant. The expression for σ_T^C coincides with that obtained in Ref. [12] by means of the Weizsäcker-Williams approximation. The accuracy of the expression (4) with σ_T^C and σ_T^{CC} given in (5) and σ_{Born} from Refs. [8, 9] is determined by the relative order of the omitted terms $\sim (Z_{A,B}\alpha)^2/L^2$. This accuracy is better than 0.4% for the RHIC and LHC colliders. In the recent papers [13, 14], the Coulomb corrections were calculated numerically for a few values of γ. We emphasize that the accuracy of the results in Refs. [13, 14] is the same as in (5). The uncertainty is related to the contribution of the region, where the energies of electron and positron are of the order of electron mass in the rest frame of one of the nuclei.

In Refs. [15, 16, 17, 18] it was claimed that the factorization of the multiple pair production probability is valid with a good accuracy, resulting in the Poisson distribution for multiplicity:

$$P_n(b) = \frac{W^n(b)}{n!}\mathrm{e}^{-W(b)}\,. \tag{6}$$

The factor $\exp(-W)$ is nothing but the vacuum-to-vacuum transition probability $P_0 = 1 - \sum_{n=1}^{\infty} P_n$. Strictly speaking, the factorization does not take place due to interference between the diagrams corresponding to the permutation of the electron (or positron) lines (see, e.g., [7]). Nevertheless, one can show that this interference gives the contribution which contains at least one power of L less than that of the amplitude squared. Therefore, in the

leading logarithmic approximation one can use the expression (6). Thus, to obtain σ_n it is sufficient to know the function $W(b)$.

In Refs. [19, 20, 21, 22, 23], the function $W_0(b)$ (the Born approximation for $W(b)$) was calculated numerically for $mb \lesssim 1$ and a few particular values of γ. The correct dependence of the function $W_0(b)$ on b at $mb \gg 1$ was obtained analytically in Ref. [24] by two different methods. Both methods give the following result:

$$W_0(b) = \frac{28}{9\pi^2}\,\frac{\zeta}{(mb)^2}\,\left[2\ln\,\gamma - 3\ln(mb)\right]\,\ln(mb)\,, \tag{7}$$

in the region $1 \ll mb \leq \sqrt{\gamma}$, and

$$W_0(b) = \frac{28}{9\pi^2}\,\frac{\zeta}{(mb)^2}\,\left(\ln\frac{\gamma}{mb}\right)^2\,, \tag{8}$$

in the region $\sqrt{\gamma} \leq mb \ll \gamma$. Note that the function $W_0(b)$ given by Eqs. (7) and (8) is the continuous function at $mb = \gamma$ together with its first derivative. Certainly, the integration of $W_0(b)$, Eqs. (7) and (8) over $\boldsymbol{b}$ gives the main term ($\propto L^3$) in σ_T^0. In the recent paper [23], the ansatz for $W_0(b)$ has been suggested, that has quite different dependence of $W_0(b)$ on γ and b for $1 \ll mb \ll \sqrt{\gamma}$. In the present paper, we confirm the result (7) once more and unambiguously disprove the ansatz suggested in Ref. [23].

The cross section σ_1 of one-pair production can be represented as follows:

$$\sigma_1 = \sigma_T + \sigma_{\rm unit} = \int d^2\,bW(b) - \int d^2\,bW(b)\left(1 - {\rm e}^{-W(b)}\right)\,. \tag{9}$$

Thus, the difference between σ_1 and σ_T is due to the unitarity correction $\sigma_{\rm unit}$. The main contribution to the term σ_T comes from $b \gg 1/m$. It was shown in Ref. [24] that the main contribution to the second term, $\sigma_{\rm unit}$, as well as the main contribution to the cross sections for the n-pair production ($n \geqslant 2$), comes from $b \sim 1/m$. As shown in Ref. [24], in this region, the function $W(b)$ has the form

$$W(b) = \zeta\,L\,\mathcal{F}(mb)\,, \tag{10}$$

where the function $\mathcal{F}(mb)$ depends on the parameters $Z_B\alpha$ and $Z_A\alpha$ and is independent of γ. Let us represent the function $\mathcal{F}(x)$ as

$$\mathcal{F}(x) = \mathcal{F}_0(x) + \mathcal{F}_A(x) + \mathcal{F}_B(x) + \mathcal{F}_{AB}(x)\,, \tag{11}$$

where $\mathcal{F}_0(x)$ is independent of Z_A and Z_B (Born term), $\mathcal{F}_A(x)$ contains terms $\propto$ $(Z_A\alpha)^{n>0}(Z_B\alpha)^0$ (Coulomb corrections with respect to the nucleus A), $\mathcal{F}_B(x)$ contains terms

$\propto (Z_A\alpha)^0(Z_B\alpha)^{n>0}$ (Coulomb corrections with respect to the nucleus B), and $\mathcal{F}_{AB}(x)$ contains terms $\propto (Z_A\alpha)^{n>0}(Z_B\alpha)^{l>0}$ (Coulomb corrections with respect to both nuclei).

In the present paper, we calculate the function $\mathcal{F}(x)$ for $Z_B\alpha \ll 1$, $Z_A\alpha \lesssim 1$, and $x \lesssim 1$. In this limit, one can neglect in Eq. (11) the terms $\mathcal{F}_B(x)$ and $\mathcal{F}_{AB}(x)$. Though $Z_B\alpha \ll 1$, we can not expand exponent in (6) if $\zeta L \sim 1$. Our method is based on the use of the quasiclassical Green's function of the Dirac equation in the Coulomb field.

II. GENERAL DISCUSSION

In the leading in $Z_B\alpha$ order, the matrix element of e^+e^- pair production, M, has the form

$$M = -e\int dt\, d\boldsymbol{r}\ \exp[-i(\varepsilon_p+\varepsilon_q)t]\overline{\Psi}_{p_-}(\boldsymbol{r})\,\hat{\mathcal{A}}(t,\,\boldsymbol{r})\,\Psi_{-p_+}(\boldsymbol{r}), \tag{12}$$

where $\mathcal{A}^\mu(t,\,\boldsymbol{r})$ is the four-vector potential of the moving nucleus B, $\overline{\Psi}_{p_-}$ and Ψ_{-p_+} are the positive- and negative-energy solutions of the Dirac equation in the Coulomb field of the nucleus A; $p_- = (\varepsilon_p,\,\boldsymbol{p})$, $p_+ = (\varepsilon_q,\,\boldsymbol{q})$ are the four-momenta of electron and positron, respectively.

Then we use the Fourier transform, $\mathcal{A}^\mu_k$, of the vector potential $\mathcal{A}^\mu(t,\,\boldsymbol{r})$,

$$\mathcal{A}^\mu_k = -\frac{4\pi e Z_B}{\boldsymbol{k}_\perp^2+(k^0/\gamma\beta)^2}e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{b}}\,2\pi\delta\left(\gamma k^0-\gamma\beta k^z\right)u^\mu\,, \tag{13}$$

where $u^\mu = (\gamma, 0, 0, \gamma\beta)$ is the four-velocity of the nucleus B, and $\boldsymbol{b}$ is the impact parameter. Taking the integrals over t, k^0, and k^z, we obtain

$$M = -\frac{4\pi Z_B\alpha}{\gamma\beta}\int\frac{d\boldsymbol{k}_\perp}{(2\pi)^2}\frac{e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{b}}}{\boldsymbol{k}_\perp^2+(E/\gamma\beta)^2}\int d\boldsymbol{r}\ \exp[i\boldsymbol{k}_\perp\cdot\boldsymbol{\rho}+iEz/\beta]\overline{\Psi}_{p_-}(\boldsymbol{r})\hat{u}\,\Psi_{-p_+}(\boldsymbol{r})\,, \tag{14}$$

where $E = \varepsilon_p+\varepsilon_q$, $\boldsymbol{r} = (\boldsymbol{\rho},\,z)$.

At the calculation of the probabilities integrated over the angles of the final particles, it is convenient to exploit the Green's functions of the Dirac equation in the external field. Using the relations (see, e.g., [25])

$$\sum_\sigma\int d\Omega_{\boldsymbol{q}}\ \Psi_{-p_+}(\boldsymbol{r}_2)\overline{\Psi}_{-p_+}(\boldsymbol{r}_1) = -i\frac{(2\pi)^2}{q\,\varepsilon_q}\delta G\left(\boldsymbol{r}_2,\boldsymbol{r}_1|-\varepsilon_q\right),$$
$$\sum_\sigma\int d\Omega_{\boldsymbol{p}}\ \Psi_{p_-}(\boldsymbol{r}_1)\overline{\Psi}_{p_-}(\boldsymbol{r}_2) = i\frac{(2\pi)^2}{p\,\varepsilon_p}\delta G\left(\boldsymbol{r}_1,\boldsymbol{r}_2|\varepsilon_p\right), \tag{15}$$

where $\delta G\left(\boldsymbol{r}, \boldsymbol{r}'|\varepsilon\right)$ is the discontinuity of the Green's function on the cut, and the summation is performed over the spin states, we obtain for the total probability:

$$
\begin{aligned}
W(b) = \sum_{\sigma\pm} |M|^2 \frac{d\boldsymbol{p}d\boldsymbol{q}}{(2\pi)^6} &= \left(\frac{2Z_B\alpha}{\gamma\beta}\right)^2 \int \frac{d\varepsilon_q d\varepsilon_p d\boldsymbol{k}_{1\perp} d\boldsymbol{k}_{2\perp}}{(2\pi)^4} \frac{\exp\left[i(\boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp})\cdot\boldsymbol{b}\right]}{[\boldsymbol{k}_{1\perp}^2 + (E/\gamma\beta)^2]\,[\boldsymbol{k}_{2\perp}^2 + (E/\gamma\beta)^2]} \\
&\times \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \, \exp\left[i\boldsymbol{k}_{2\perp}\cdot\boldsymbol{\rho}_2 - i\boldsymbol{k}_{1\perp}\cdot\boldsymbol{\rho}_1 + i\frac{E}{\beta}(z_2 - z_1)\right] \\
&\times \mathrm{Sp}\left[\hat{u}\,\delta G(\boldsymbol{r}_2, \boldsymbol{r}_1| - \varepsilon_q)\,\hat{u}\;\delta G(\boldsymbol{r}_1, \boldsymbol{r}_2\,|\varepsilon_p)\right] .
\end{aligned} \tag{16}
$$

Using the gauge invariance and the condition $\gamma \gg 1$, it is possible to make in Eq. (16) the following replacement

$$
\mathrm{Sp}\left[\hat{u}\,\delta G(\boldsymbol{r}_2, \boldsymbol{r}_1| - \varepsilon_q)\,\hat{u}\;\delta G(\boldsymbol{r}_1, \boldsymbol{r}_2\,|\varepsilon_p)\right] \to \frac{\gamma^2}{E^2}\mathrm{Sp}\left[\hat{k}_{2\perp}\,\delta G(\boldsymbol{r}_2, \boldsymbol{r}_1| - \varepsilon_q)\,\hat{k}_{1\perp}\;\delta G(\boldsymbol{r}_1, \boldsymbol{r}_2\,|\varepsilon_p)\right] . \tag{17}
$$

In the leading logarithmic approximation, the main contribution to the probability $W(\boldsymbol{b})$ comes from the region $\varepsilon_\pm \gg m$, where the quasiclassical approximation is applicable. Besides, it is convenient to perform the calculations in terms of the Green's function $D(\boldsymbol{r}, \boldsymbol{r}'|\varepsilon)$ of the squared Dirac equation [25, 26]. Using the transformations similar to those in Ref. [26], we obtain

$$
\begin{aligned}
W(b) = 4\left(Z_B\alpha\right)^2 &\int \frac{d\varepsilon_q d\varepsilon_p d\boldsymbol{k}_{1\perp} d\boldsymbol{k}_{2\perp}}{E^2\;(2\pi)^4} \frac{\exp\left[i(\boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp})\cdot\boldsymbol{b}\right]}{[\boldsymbol{k}_{1\perp}^2 + (E/\gamma\beta)^2]\,[\boldsymbol{k}_{2\perp}^2 + (E/\gamma\beta)^2]} \\
&\times \int d\boldsymbol{r}_1 d\boldsymbol{r}_2 \, \exp\left[i\boldsymbol{k}_{2\perp}\cdot\boldsymbol{\rho}_2 - i\boldsymbol{k}_{1\perp}\cdot\boldsymbol{\rho}_1 + i\frac{E}{\beta}(z_2 - z_1)\right] \\
&\times \mathrm{Sp}\left\{\left[\left[-2i\boldsymbol{k}_{2\perp}\cdot\boldsymbol{\nabla}_2 + \hat{k}_2\hat{k}_{2\perp}\right] D(\boldsymbol{r}_2, \boldsymbol{r}_1| - \varepsilon_q)\right]\right. \\
&\times \left.\left[\left[-2i\boldsymbol{k}_{1\perp}\cdot\boldsymbol{\nabla}_1 - \hat{k}_1\hat{k}_{1\perp}\right]\; D(\boldsymbol{r}_1, \boldsymbol{r}_2\,|\varepsilon_p)\right]\right\} .
\end{aligned} \tag{18}
$$

Here $k_1 = (E, \boldsymbol{k}_{1\perp}, E)$, $k_2 = (E, \boldsymbol{k}_{2\perp}, E)$. In the quasiclassical approximation, the function D has the form [25]

$$
\begin{aligned}
D(\boldsymbol{r}_2, \boldsymbol{r}_1|\varepsilon) \;=\; &\frac{i\kappa \mathrm{e}^{i\kappa r}}{8\pi^2 r_1 r_2} \int d\boldsymbol{q} \exp\left[i\,\frac{\kappa r(\boldsymbol{q} + \boldsymbol{f})^2}{2r_1 r_2}\right] \left(\frac{4r_1 r_2}{q^2}\right)^{iZ_A\alpha\lambda} \\
&\times \left[1 + \frac{\lambda r}{2r_1 r_2}\boldsymbol{\alpha}\cdot(\boldsymbol{q} + \boldsymbol{f})\right] ,
\end{aligned}
$$

$$
\kappa = \sqrt{\varepsilon^2 - m^2} \quad , \qquad \lambda = \frac{\varepsilon}{\kappa}\,, \quad \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}\,, \quad \boldsymbol{f} = \frac{[[\boldsymbol{r}_1 \times \boldsymbol{r}_2] \times \boldsymbol{r}]}{r^2}\,, \quad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2\,, \tag{19}
$$

where $\boldsymbol{q}$ is a two-dimensional vector lying in the plane perpendicular to $\boldsymbol{r}$. The explicit form (19) of the quasiclassical Green's function is very convenient for analytical investigation of high-energy processes in the Coulomb field.

III. ANALYTICAL RESULTS

For $mb \lesssim 1$, the main contribution to the integrals in Eq. (18) is given by the region of small angles between vectors $\boldsymbol{r}_1$, $-\boldsymbol{r}_2$, and z-axis. Using these conditions and the quasiclassical Green's function (19), we obtain the following representation for $\mathcal{F}(mb) = \mathcal{F}_0(mb) + \mathcal{F}_A(mb)$ (details of the calculation are presented in Appendix)

$$
\begin{aligned}
\mathcal{F}(mb) = & \frac{1}{\pi^4 \left(Z_A\alpha\right)^2} \int_0^1 dx \int d^2Q \int \frac{d^2\beta}{\boldsymbol{\beta}^2} \left[1 - \left(\frac{|\boldsymbol{R} + x\boldsymbol{Q}|}{|\boldsymbol{R} - \bar{x}\boldsymbol{Q}|}\right)^{2iZ_A\alpha}\right] \\
& \times \left\langle 4\sqrt{x\bar{x}}\,(x-\bar{x})\,\boldsymbol{\beta}\cdot\boldsymbol{Q}\left(K_1^2(\tilde{Q})/\tilde{Q}^2 - K_1(Q)K_1(\tilde{Q})/Q\tilde{Q}\right) + \left[K_0(\tilde{Q}) - K_0(Q)\right]^2 \right. \\
& \left. + 4x\bar{x}\beta^2 K_1^2(\tilde{Q})/\tilde{Q}^2 + \left(Q^2 - 4x\bar{x}\left(\boldsymbol{\beta}\cdot\boldsymbol{Q}\right)^2/\beta^2\right)\left[K_1(\tilde{Q})/\tilde{Q} - K_1(Q)/Q\right]^2 \right\rangle, \\
& \tilde{Q}^2 = Q^2 + \beta^2\,, \quad \boldsymbol{R} = \sqrt{x\bar{x}}\boldsymbol{\beta} + m\boldsymbol{b}\,, \quad \bar{x} = 1 - x\,,
\end{aligned}
\tag{20}
$$

where $K_n(x)$ is a modified Bessel function of the third kind. The form (20) is suitable for investigation of the asymptotics of $\mathcal{F}(mb)$. For numerical evaluation, it is convenient to pass from the integration over the angle ϕ of the vector $\boldsymbol{Q}$ to the integration over the parameter v using the following identities

$$
\begin{aligned}
& \int \frac{d\phi}{2\pi}\left[1 - \left(\frac{1 + a\,\cos\phi}{1 - b\,\cos\phi}\right)^{i\nu}\right]\begin{Bmatrix} 1 \\ \cos\phi \\ \cos 2\phi \end{Bmatrix} \\
& = \frac{\nu\,\sinh\pi\nu}{\pi}\lim_{\delta\to 0}\int_0^1 \frac{dv}{v^{1-\delta}\,\bar{v}^{1-\delta}}\left(\frac{v}{\bar{v}}\right)^{-i\nu}\begin{Bmatrix} \ln\frac{1+\sqrt{1-s^2}}{2} \\ \frac{s}{1+\sqrt{1-s^2}} \\ -\frac{1}{2}\left(\frac{s}{1+\sqrt{1-s^2}}\right)^2 \end{Bmatrix} \\
& \bar{v} = 1 - v\,, \quad s = a\,v - b\,\bar{v}\,.
\end{aligned}
\tag{21}
$$

Making the substitution $v = u/(u + \xi\bar{u})$, where

$$
\bar{u} = 1 - u\,, \quad \xi = \frac{R^2 + \bar{x}^2Q^2}{R^2 + x^2Q^2}\,,
$$

and taking into account the symmetry of the integrand with respect to the substitution $u \to \bar{u}$, $x \to \bar{x}$, we obtain

$$
\begin{aligned}
\mathcal{F}(mb) = {} & 4\frac{\sinh(\pi Z_A\alpha)}{\pi^4 Z_A\alpha}\int_0^1 dx \int_0^\infty dQ\, Q \int \frac{d^2\beta}{\beta^2}\int_0^{1/2}\frac{du}{u\,\bar{u}}\cos\left[Z_A\alpha\ln\left(u/\bar{u}\right)\right] \\
& \times\left\langle \ln\left(\frac{s+\sqrt{s^2-t^2}}{g}\right)\left\{(1-2x\bar{x})\left[QK_1(\tilde{Q})/\tilde{Q}-K_1(Q)\right]^2\right.\right. \\
& \left. + \left[K_0(\tilde{Q})-K_0(Q)\right]^2 + 4x\bar{x}K_1^2(\tilde{Q})\beta^2/\tilde{Q}^2\right\} + x\bar{x}\left[2\left(\frac{\boldsymbol{\beta}\cdot\boldsymbol{R}}{\beta R}\right)^2-1\right] \\
& \times\left[\left(\frac{t}{s+\sqrt{s^2-t^2}}\right)^2-\left(\frac{RQ\bar{x}}{g}\right)^2\right]\left[QK_1(\tilde{Q})/\tilde{Q}-K_1(Q)\right]^2 \\
& \left. -4\sqrt{x\bar{x}}\,(\bar{x}-x)\,\frac{\boldsymbol{\beta}\cdot\boldsymbol{R}}{R}\left[\frac{t}{s+\sqrt{s^2-t^2}}+\frac{RQ\bar{x}}{g}\right]\left[QK_1^2(\tilde{Q})/\tilde{Q}^2-K_1(Q)K_1(\tilde{Q})\tilde{Q}\right]\right\rangle, \\
& \tilde{Q}^2 = Q^2+\beta^2\,, \quad g=\max(R^2, Q^2\bar{x}^2)\,, \\
& t = 2QR\,(xu-\bar{x}\bar{u})\;, \quad s = R^2+Q^2\left(x^2u+\bar{x}^2\bar{u}\right).
\end{aligned}
\tag{22}
$$

Let us consider the asymptotics of Eq. (20). For $mb \gg 1$, there are two regions of integration over $\boldsymbol{\beta}$, giving the leading logarithmic contribution to $\mathcal{F}(mb)$: $1 \ll |\sqrt{x\bar{x}}\boldsymbol{\beta} + m\boldsymbol{b}| \ll mb$ and $1 \ll \beta \ll mb$. These regions give equal contributions, and the final result reads

$$
\mathcal{F}(mb) = \frac{56}{9\pi^2(mb)^2}\ln(mb)\,. \tag{23}
$$

Thus, the leading logarithmic contribution is given by the Born term $\mathcal{F}_0(mb)$. This asymptotics agrees with Eq. (7) under the condition $\ln(mb) \ll L$.

The main contribution to $\mathcal{F}_A(mb)$ comes from the region $|\sqrt{x\bar{x}}\boldsymbol{\beta} + m\boldsymbol{b}| \sim 1$ and has the form

$$
\mathcal{F}_A(mb) = -\frac{28}{9\pi^2(mb)^2}f(Z_A\alpha)\,, \tag{24}
$$

where the function $f(x)$ is defined in Eq. (5). Again, this asymptotics is valid under the condition $\ln(mb) \ll L$. Similar to the derivation of Eq. (7), see Ref. [24], based on the equivalent photon approximation, it is possible to obtain for the Coulomb corrections to $W(b)$ with respect to the nucleus A, $W_A(b)$, the expression valid in the wider region $\ln(mb) \lesssim L$ (but still $1 \ll mb \ll \gamma$). We have

$$
W_A(b) = -\frac{28}{9\pi^2}\frac{\zeta}{(mb)^2}f(Z_A\alpha)\ln\left(\frac{\gamma}{mb}\right)\,. \tag{25}
$$

Eq. (24) evidently agrees with Eq. (25).

Let us consider the asymptotics at small impact parameters. For $mb \ll 1$, the leading logarithmic contribution comes from the region $mb \ll \beta \sim Q \ll 1$. Taking the integrals over this region, we obtain

$$\begin{aligned}\mathcal{F}(mb) =& \frac{8}{3\pi^2 (Z_A\alpha)^2} \ln\left(\frac{1}{mb}\right) \mathrm{Re}\left[\psi(1+iZ_A\alpha) + C - (Z_A\alpha)^2\right.\\ &\left.+ iZ_A\alpha(1+(Z_A\alpha)^2)\psi'(1+iZ_A\alpha)\right] .\end{aligned} \tag{26}$$

This asymptotics is obtained for zero nuclear radius R_n. In order to obtain $W(b)$ for the extended nuclei, it is sufficient, within the logarithmic accuracy, to make the substitution $\ln(mb) \to \ln(mb + mR_n)$ in the asymptotics (26). For $b \gg R_n$, the finite-nuclear-size correction to $W(b)$ is negligible.

IV. NUMERICAL RESULTS

Using Eq. (18), we performed the tabulation of the function $\mathcal{F}(mb)$ for a few values of Z_A. The corresponding results are presented on the left plot of Fig. 1 and in the Table I. We remind that these results are obtained in the Born approximation with respect to the nucleus B. For most experiments $Z_A = Z_B$, and it is necessary to know the function $\mathcal{F}(mb)$ beyond the Born approximation with respect to the nucleus B. If we assume that the term $\mathcal{F}_{AB}$ in Eq. (11) is numerically small, then we can approximate the function $\mathcal{F}$ as $\mathcal{F}_0 + 2\mathcal{F}_A$ in this case. This function is shown on the right plot of Fig. 1. It is seen that the Coulomb corrections in the region $mb \lesssim 1$ are very important for the experimentally interesting case $Z_A = Z_B = 79$. The assumption of smallness of the contribution $\mathcal{F}_{AB}$ is supported by the comparison of our results for $W(b)$ with those obtained in Refs. [14, 27] for $Z_A = Z_B = 79$ and $\gamma = 2 \times 10^4$ ($\gamma_{c.m.} = 100$).

As we already pointed out, Eq. (10) has logarithmic accuracy which can be sufficient for very large γ. In order to go beyond the logarithmic accuracy, we represent $W(b)$ in the form

$$W(b) = \zeta\left[L - G(mb)\right]\mathcal{F}(mb)\,, \tag{27}$$

where $G(mb)$ is some function of mb and, generally speaking, of the parameters $Z_A\alpha$ and $Z_B\alpha$. The asymptotics of $G(mb)$ at $1 \ll mb \ll \sqrt{\gamma}$ is known, see Eqs. (7) and (25). However, the calculation of the function $G(mb)$ at $mb \lesssim 1$ is rather complicated problem.

Instead, we use the results of numerical calculations, performed for a few values of γ in Refs. [19, 27] in the Born approximation. We have found that the form

$$G(mb) = \frac{3}{2}\ln(mb+1) + 1.9\,, \tag{28}$$

provides good agreement of Eq. (27) with the numerical results of Refs. [19, 27, 28] in the wide region of mb, see Fig. 2. The form (28) of $G(mb)$ is obtained by fitting the Born results and thus is independent of $Z_{A,B}$. It provides the correct asymptotics of $W_0(b)$, Eq. (7). It turns out that the formula (27) with $G(mb)$ from Eq. (28) has also high accuracy for $Z_A\alpha$, $Z_B\alpha \lesssim 1$ in the region $mb \lesssim 1$ where the Coulomb corrections are large. We have checked this fact by comparing our results with those of Ref. [27] obtained numerically for $Z_A = Z_B = 79$, see Fig. 2. Note that the tabulation of $W(b)$ and $P_N(b)$, performed in Refs.[13, 14, 19, 27, 28] for a few values of γ, required the evaluation of nine-fold integral and, therefore, was very laborious. The calculation of the function $\mathcal{F}$ from Eq. (22) is essentially simpler. Besides, since this function is independent of γ, one can easily obtain predictions for $W(b)$ at any $\gamma \gg 1$ using Eqs. (27) and (28).

V. CONCLUSION

In the present paper, we have found in the leading logarithmic approximation the simple representation for the function $W(b)$ for $mb \lesssim 1$, $Z_B\alpha \ll 1$, and arbitrary $Z_A\alpha$. Using the results of numerical calculation of $W(b)$ performed for a few values of γ and $Z_{A,B}$, we have obtained the approximate formula for $W(b)$ valid in a wide region of parameters: $mb \lesssim \sqrt{\gamma}$, $Z_A\alpha \lesssim 1$, $Z_B\alpha \lesssim 1$, $\gamma \gg 1$. We estimate the accuracy of this formula to be a few percent. The results obtained clearly demonstrate the dependence of $W(b)$, as well as $P_n(b)$, on the relativistic factor γ and the parameters $Z_{A,B}\alpha$.

This work was supported in part by RFBR grant No. 05-02-16079 and by the grant for young scientists of SB RAS (R.N.L.).

APPENDIX A: CALCULATION OF THE INTEGRALS

In this Appendix, we present some details of derivation of Eq. (20) from Eq. (18). The main contribution to the integrals comes from the region of small angles between vectors $\boldsymbol{r}_1$,

x	Born	Au	Pb	U	x	Born	Au	Pb	U
0.01	3.42	2.76	2.71	2.56	1.26	0.391	0.347	0.343	0.332
0.0126	3.26	2.65	2.59	2.45	1.58	0.304	0.273	0.27	0.262
0.0158	3.11	2.52	2.47	2.34	2.	0.231	0.209	0.207	0.202
0.02	2.96	2.4	2.35	2.22	2.51	0.171	0.156	0.155	0.152
0.0251	2.8	2.28	2.24	2.11	3.16	0.124	0.114	0.114	0.111
0.0316	2.65	2.16	2.12	2.	3.98	8.78×10^{-2}	8.2×10^{-2}	8.15×10^{-2}	8.01×10^{-2}
0.0398	2.5	2.04	2.0	1.89	5.01	6.14×10^{-2}	5.78×10^{-2}	5.75×10^{-2}	5.66×10^{-2}
0.0501	2.34	1.92	1.88	1.78	6.31	4.25×10^{-2}	4.02×10^{-2}	4.0×10^{-2}	3.95×10^{-2}
0.0631	2.19	1.8	1.76	1.67	7.94	2.91×10^{-2}	2.77×10^{-2}	2.76×10^{-2}	2.73×10^{-2}
0.0794	2.04	1.68	1.64	1.56	10	1.99×10^{-2}	1.9×10^{-2}	1.89×10^{-2}	1.87×10^{-2}
0.1	1.88	1.55	1.52	1.45	12.6	1.35×10^{-2}	1.29×10^{-2}	1.29×10^{-2}	1.28×10^{-2}
0.126	1.73	1.43	1.41	1.34	15.8	9.07×10^{-3}	8.75×10^{-3}	8.72×10^{-3}	8.64×10^{-3}
0.158	1.58	1.31	1.29	1.23	20	6.09×10^{-3}	5.89×10^{-3}	5.87×10^{-3}	5.83×10^{-3}
0.2	1.43	1.19	1.17	1.12	25.1	4.07×10^{-3}	3.95×10^{-3}	3.94×10^{-3}	3.91×10^{-3}
0.251	1.28	1.07	1.06	1.01	31.6	2.71×10^{-3}	2.64×10^{-3}	2.63×10^{-3}	2.61×10^{-3}
0.316	1.14	0.961	0.941	0.898	39.8	1.8×10^{-3}	1.75×10^{-3}	1.75×10^{-3}	1.74×10^{-3}
0.398	0.993	0.842	0.829	0.793	50.1	1.19×10^{-3}	1.16×10^{-3}	1.16×10^{-3}	1.15×10^{-3}
0.501	0.856	0.731	0.72	0.69	63.1	7.9×10^{-4}	7.71×10^{-4}	7.69×10^{-4}	7.65×10^{-4}
0.631	0.725	0.625	0.616	0.591	79.4	5.21×10^{-4}	5.09×10^{-4}	5.08×10^{-4}	5.05×10^{-4}
0.794	0.603	0.524	0.517	0.498	100	3.43×10^{-4}	3.35×10^{-4}	3.34×10^{-4}	3.33×10^{-4}
1.	0.491	0.431	0.426	0.411					

TABLE I: The function $\mathcal{F}(x)$, Eq. (22), calculated in the Born approximation ($Z_A\alpha \to 0$) and exactly in the parameter $Z_A\alpha$ for Au, Pb, and U.

$-\boldsymbol{r}_2$, and z-axis. Using this fact, we take the integrals over the angles of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, make the substitution $r_{1,2} \to E\, r_{1,2}$, and change the variables $\varepsilon_p = Ex$, $\varepsilon_q = E\bar{x} = E(1-x)$. Taking the integral over E in the logarithmic approximation at $\gamma \gg 1$ and $mb \lesssim 1$, we obtain

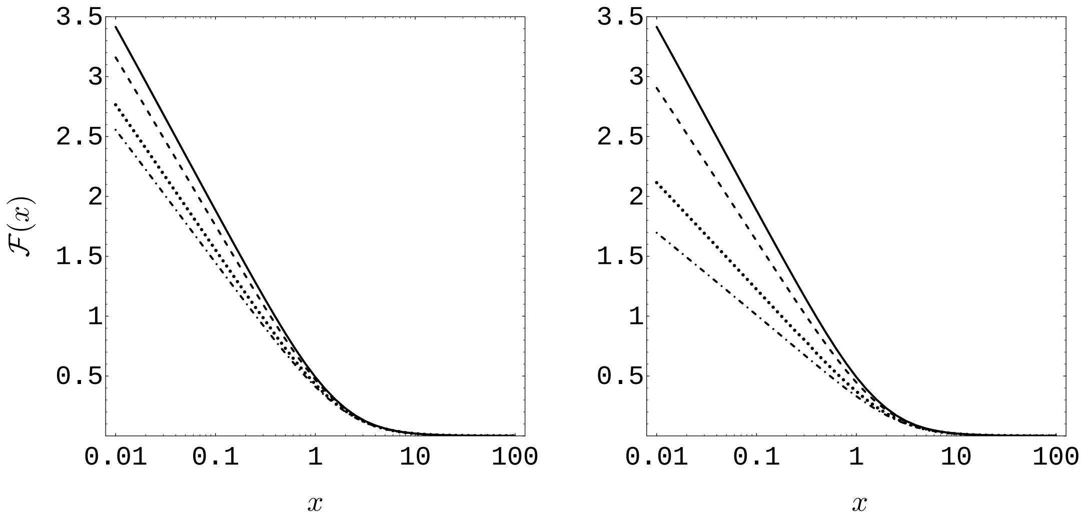

FIG. 1: The function $\mathcal{F}(x)$, Eq. (10), for $Z_A = 92$ (dash-dotted line) , $Z_A = 79$ (dotted line), $Z_A = 47$ (dashed line), and the Born approximation (solid line). Left plot corresponds to the Born approximation in $Z_B\alpha$, Eq. (22). Right plot shows the results obtained from Eqs. (11) and (22) for $Z_B = Z_A$ with the term $\mathcal{F}_{AB}(x)$ omitted.

$$
\begin{aligned}
dW(b) &= \frac{(Z_B\alpha)^2}{(2\pi)^6}\ln\gamma \int \frac{d\boldsymbol{k}_{1\perp}}{k_{1\perp}^2}\frac{d\boldsymbol{k}_{2\perp}}{k_{2\perp}^2}\int dx\, x\,\bar{x}\frac{dr_1}{r_1}\frac{dr_2}{r_2}\int d\boldsymbol{Q}\, d\boldsymbol{q}\left(\frac{|\boldsymbol{q}+\boldsymbol{Q}|}{|\boldsymbol{q}-\boldsymbol{Q}|}\right)^{2iZ_A\alpha} \\
&\times \exp\left[-\frac{i}{2}m^2(r_1+r_2) - i\,\boldsymbol{\Delta}\cdot\boldsymbol{\beta} - \frac{i}{2}x\bar{x}(r_1k_{1\perp}^2 + r_2k_{2\perp}^2) + \frac{i(r_1+r_2)\boldsymbol{Q}^2}{2r_1r_2}\right] \\
&\times \left\langle 2\,(\bar{x}-x)\left(\frac{k_{1\perp}^2\boldsymbol{k}_{2\perp}\cdot\boldsymbol{Q}}{r_2} - \frac{k_{2\perp}^2\boldsymbol{k}_{1\perp}\cdot\boldsymbol{Q}}{r_1}\right) - 4x\bar{x}k_{1\perp}^2k_{2\perp}^2 - \frac{4\,(\boldsymbol{k}_{1\perp}\cdot\boldsymbol{Q})\,(\boldsymbol{k}_{2\perp}\cdot\boldsymbol{Q})}{r_1r_2}\right. \\
&\left. -\,(\boldsymbol{k}_{1\perp}\cdot\boldsymbol{k}_{2\perp})\left(\frac{m^2(r_1+r_2)^2}{2x\bar{x}r_1r_2} + \frac{r_1k_{1\perp}^2}{2r_2} + \frac{r_2k_{2\perp}^2}{2r_1}\right)\right\rangle, \\
\boldsymbol{\Delta} &= \boldsymbol{k}_{1\perp} - \boldsymbol{k}_{2\perp}, \quad \beta = \boldsymbol{q}/2 + (\bar{x}-x)\,\boldsymbol{Q}/2 - \boldsymbol{b}\,. \qquad \text{(A1)}
\end{aligned}
$$

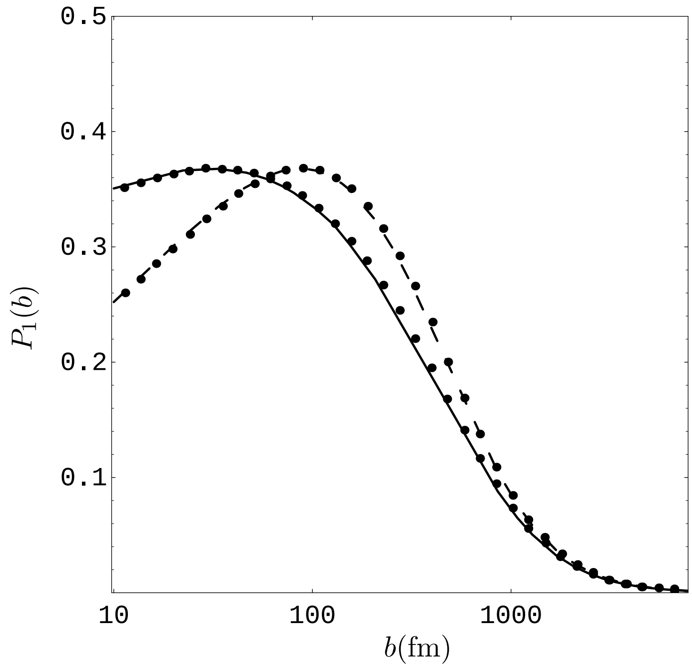

FIG. 2: The probability of one-pair production $P_1(b)$ corresponding to the function $W(b)$ from Eq. (27), $\gamma = 2\times 10^4$, $Z_A = Z_B = 79$. Dashed line: the function $\mathcal{F}$ is taken in the Born approximation, $\mathcal{F} = \mathcal{F}_0$; solid line: the Coulomb corrections are taken into account, $\mathcal{F} = \mathcal{F}_0 + 2\mathcal{F}_A$. Dots show the corresponding results of numerical calculations from Ref. [27].

The integration over two-dimensional vectors $\boldsymbol{k}_{1\perp}$ and $\boldsymbol{k}_{2\perp}$ can be easily performed. The result reads

$$dW(\boldsymbol{b}) = -\frac{(Z_B\alpha)^2}{(2\pi)^4}\ln\gamma \int dx\,\frac{dr_1}{r_1^2}\frac{dr_2}{r_2^2}\int d\boldsymbol{Q}\,d\boldsymbol{q}\exp\left[-\frac{i}{2}m^2(r_1+r_2)+\frac{i(r_1+r_2)\boldsymbol{Q}^2}{2r_1r_2}\right]$$
$$\times\left(\frac{|\boldsymbol{q}+\boldsymbol{Q}|}{|\boldsymbol{q}-\boldsymbol{Q}|}\right)^{2iZ_A\alpha}\left\langle\left[2\,(\bar{x}-x)\,\frac{\boldsymbol{\beta}\cdot\boldsymbol{Q}}{\boldsymbol{\beta}^2}+\frac{1}{2x\bar{x}}\right](2\mathcal{E}_1\mathcal{E}_2-\mathcal{E}_1-\mathcal{E}_2)-4\mathcal{E}_1\mathcal{E}_2\right.$$
$$\left.+\left[\frac{4\,(\boldsymbol{\beta}\cdot\boldsymbol{Q})^2}{\boldsymbol{\beta}^4}x\bar{x}+\frac{m^2(r_1+r_2)^2}{2\boldsymbol{\beta}^2}\right](\mathcal{E}_1-1)\,(\mathcal{E}_2-1)\right\rangle, \tag{A2}$$

$\mathcal{E}_i = \exp[i\boldsymbol{\beta}^2/2x\bar{x}r_i]$

Taking the integrals over $r_{1,2}$ and passing from the variable $\boldsymbol{q}$ to $\boldsymbol{\beta} = \boldsymbol{q}/2 + (\bar{x} - x)\,\boldsymbol{Q}/2 - \boldsymbol{b}$, we obtain Eq. (20).

---

[1] C. Bertulani and G. Baur, Phys. Rep. **163**, 299 (1988).

[2] G. Baur, K. Hencken, D. Trautmann, S. Sadovsky, Yu. Kharlov, Phys. Rep. **364**, 359–450 (2002).

[3] C.A. Bertulani, S.R. Klein, and J. Nystrand, Annu. Rev. Nucl. Part. Sci. **55**, 271 (2005).

[4] B. Segev and J.C. Wells, Phys. Rev. A **57**, 1849 (1998); physics/9805013.

[5] A.J. Baltz and L. McLerran, Phys. Rev. C **58**, 1679 (1998).

[6] U. Eichmann, J. Reinhardt, S. Schramm, and W. Greiner, Phys. Rev. A **59**, 1223 (1999).

[7] A.J. Baltz, F. Gelis, L. McLerran, and A. Peshier, Nucl. Phys. A **695**, 395 (2001).

[8] L.D. Landau and E.M. Lifshitz, Phys. Z. Sowjet. **6**, 244 (1934).

[9] G. Racah, Nuovo Cim. **14**, 93 (1937).

[10] R.N. Lee, A.I. Milstein, Phys. Rev. A. **61**, 032103 (2000).

[11] R.N. Lee and A.I. Milstein, Phys. Rev. A **64**, 032106 (2001).

[12] D.Yu. Ivanov, A. Schiller, and V.G. Serbo, Phys. Lett. B **454**, 155 (1999).

[13] A.J. Baltz, Phys. Rev. C **71**, 024901 (2005), Erratum-ibid. C **71**, 039901 (2005).

[14] A.J. Baltz, arxiv: nucl-th/0608006.

[15] G. Baur, Phys. Rev. D **41**, 3535 (1990).

[16] M.J. Rhoades-Brown and J. Weneser, Phys. Rev. A **44**, 330 (1991).

[17] C. Best, W. Greiner, and G. Soff, Phys. Rev. A **46**, 261 (1992).

[18] K. Hencken, D. Trautmann, and G. Baur. Phys. Rev. A **51**, 998 (1995).

[19] K. Hencken, D. Trautmann, G. Baur. Phys. Rev. A **51**, 1874 (1995).

[20] M.C. Güçlü, J.C. Wells, A.S. Umar, M.R. Strayer, and D.J. Ernst, Phys. Rev. A **51**, 1836 (1995).

[21] M.C. Güçlü, Nucl. Phys. A **668**, 149 (2000).

[22] K. Hencken, G. Baur, D. Trautmann, Phys. Rev. C **69**, 054902 (2004).

[23] M.C. Güçlü, G. Kovankaya, and M. Yilmaz, Phys. Rev. A **72**, 022724 (2005).

[24] R.N. Lee, A.I. Milstein, and V.G. Serbo, Phys. Rev. A **65**, 022102 (2002).

[25] R.N. Lee, A.I. Milstein, and V.M. Strakhovenko, Phys. Rev. A **69**, 022708 (2004).

[26] R.N. Lee, A.I. Milstein, V.M. Strakhovenko, and O.Ya. Schwartz, JETP **100**, 1 (2005).

[27] K. Hencken, D. Trautmann, G.Baur, Phys. Rev. C **59**, 841 (1999).

[28] A. Alscher, K. Hencken, D. Trautmann, G.Baur, Phys. Rev. A **55**, 396 (1997).